

# ČEBYŠEV SUBSPACES OF JBW\*-TRIPLES

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**ABSTRACT.** We describe the one-dimensional Čebyšev subspaces of a JBW\*-triple  $M$ , by showing that for a non-zero element  $x$  in  $M$ ,  $\mathbb{C}x$  is a Čebyšev subspace of  $M$  if, and only if,  $x$  is a Brown-Pedersen quasi-invertible element in  $M$ . We study the Čebyšev JBW\*-subtriples of a JBW\*-triple  $M$ . We prove that, for each non-zero Čebyšev JBW\*-subtriple  $N$  of  $M$ , then exactly one of the following statements holds:

- (a)  $N$  is a rank one JBW\*-triple with  $\dim(N) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $N$  may be a closed subspace of arbitrary dimension and  $M$  may have arbitrary rank;
- (b)  $N = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $M$ ;
- (c)  $N$  and  $M$  have rank two, but  $N$  may have arbitrary dimension;
- (d)  $N$  has rank greater or equal than three and  $N = M$ .

We also provide new examples of Čebyšev subspaces of classic Banach spaces in connection with ternary rings of operators.

## 1. INTRODUCTION

Let  $V$  be a subspace of a Banach space  $X$ . The subspace  $V$  is called a *Čebyšev (Chebyshev) subspace* of  $X$  if and only if for each  $x \in X$  there exists a unique point  $x_\circ \in V$  such that  $\text{dist}(x, V) = \|x - x_\circ\|$ .

Let  $K$  be a compact Hausdorff space. A classical theorem due to A. Haar establishes that an  $n$ -dimensional subspace  $V$  of the space  $C(K)$ , of all continuous complex-valued functions on  $K$ , is a Čebyšev subspace of  $C(K)$  if, and only if, any non-zero  $f \in V$  admits at most  $n - 1$  zeros (cf. [19] and the monograph [33, p. 215]). Having in mind the Riesz representation theorem, and the characterization of the extreme points of the closed unit ball in the dual space of  $C(K)$ , we can easily see that, in the above conditions,  $V$  is an  $n$ -dimensional Čebyšev subspace of  $C(K)$  if, and

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only if, for every set  $\{\delta_{t_1}, \dots, \delta_{t_n}\}$  of  $n$ -mutually orthogonal pure states we have  $V \cap \bigcap_{i=1}^n \ker(\delta_{t_i}) = \{0\}$ . This result implies that any non-zero  $f$  in  $C(K)$  spans a Čebyšëv subspace of the latter space if, and only if,  $f$  is invertible in the algebra  $C(K)$ .

Later on, J.G. Stampfli proved in [34, Theorem 2], that the scalar multiples of the unit element in a von Neumann algebra  $M$  is a Čebyšëv subspace of  $M$ . In [26], D.A. Legg, B.E. Scranton, and J.D. Ward characterize the semi-Čebyšëv and finite dimensional Čebyšëv subspaces of  $K(H)$ , the algebra of compact operators on an infinite-dimensional Hilbert space  $H$ . They conclude that, for a separable Hilbert space  $H$ , there exist Čebyšëv subspaces of every finite dimension in  $K(H)$  [26, Theorem 3], when  $H$  is not separable  $K(H)$  has no finite-dimensional Čebyšëv subspaces [26, Corollary 2].

A.G. Robertson continued with the study on Čebyšëv subspaces of von Neumann algebras in [30], where he established the following results:

**Theorem 1.** ([30, Theorem 6]) *Let  $x$  be a non-zero element in a von Neumann algebra  $M$ . Then, the one dimensional subspace  $\mathbb{C}x$  is a Čebyšëv subspace of  $M$  if and only if there is a projection  $p$  in the center of  $M$  such that  $px$  is left invertible in  $pM$  and  $(1-p)x$  is right invertible in  $(1-p)M$ .*

**Theorem 2.** ([30, Theorem 6]) *Let  $N$  be finite dimensional  $*$ -subalgebra of an infinite dimensional von Neumann algebra  $M$ . Suppose  $N$  has dimension  $> 1$ . Then  $N$  is not a Čebyšëv subspace of  $M$ .*

A.G. Robertson and D. Yost prove in [31, Corollary 1.4] that an infinite dimensional  $C^*$ -algebra  $A$  admits a finite dimensional  $*$ -subalgebra  $B$  which is also a Čebyšëv in  $A$  if and only if  $A$  is unital and  $B = \mathbb{C}1$ .

The results proved by Robertson and Yost were complemented by G.K. Pedersen, who shows that if  $A$  is a  $C^*$ -algebra without unit and  $B$  is a Čebyšëv  $C^*$ -subalgebra of  $A$ , then  $A = B$  (compare [29, Theorem 4]).

The previous results of Robertson [30] and Pedersen [29, Theorem 2] also prove the following equivalent reformulation of Theorem 1: for each non-zero element  $x$  in a von Neumann algebra  $M$ , the following statements are equivalent:

- (a)  $\mathbb{C}x$  is a Čebyšëv subspace of  $M$ ;
- (b)  $x$  is Brown-Pedersen quasi-invertible in  $M$ ;
- (c) For each pure state (i.e. for each extreme point of the positive part of the closed unit ball of  $M^*$ )  $\varphi \in M^*$ , and for each unitary  $u \in M$ , we have  $\varphi(x^*x) + \varphi(uxx^*u) > 0$ .

Then, the one dimensional subspace  $\mathbb{C}x$  is a Čebyšëv subspace of  $M$  if and only if there is a projection  $p$  in the center of  $M$  such that  $px$  is left invertible in  $pM$  and  $(1-p)x$  is right invertible in  $(1-p)M$ .

A renewed interest on Čebyšev subspaces of  $C^*$ -algebras has led M. Namboodiri, S. Pramod, and A. Vijayarajan to revisit and generalize the previous contributions of Robertson, Yost and Pedersen in [28].

On the other hand,  $C^*$ -algebras can be regarded as elements in a strictly wider class of complex Banach spaces called JBW\*-triples (see §2 for the detailed definitions). Many geometric properties studied in the setting of  $C^*$ -algebras have been also explored in the bigger class of JBW\*-triples. However Čebyšev subspaces and the theory of best approximations remains unexplored in the class of JBW\*-triples. In this note we present the first results about Čebyšev subspaces and Čebyšev subtriples in Jordan structures.

In Section 2 we prove that for a non-zero element  $x$  in a JBW\*-triple  $M$ ,  $\mathbb{C}x$  is a Čebyšev subspace of  $M$  if, and only if,  $x$  is a Brown-Pedersen quasi-invertible element in  $M$  (see Theorem 6). This result generalizes the result established by Robertson in Theorem 1 (cf. [30]), but it also add a new perspective from an independent argument.

In Section 3 we establish a precise description of the JBW\*-subtriples of a JBW\*-triple  $M$  which are Čebyšev subspaces in  $M$ . We should remark that in the setting of von Neumann algebras and  $C^*$ -algebras, the scarcity of non-trivial Čebyšev \*-subalgebras is endorsed with the following results: If an infinite dimensional von Neumann algebra,  $M$ , contains a finite dimensional von Neumann subalgebra  $N$  which is a Čebyšev subspace in  $M$ , then  $N$  must be one dimensional (compare Theorem 2 or [30, Theorem 6]). Furthermore, an infinite dimensional  $C^*$ -algebra  $A$  admits a finite dimensional \*-subalgebra  $B$  which is also a Čebyšev in  $A$  if and only if  $A$  is unital and  $B = \mathbb{C}1$  (cf. [31, Corollary 1.4]). If  $A$  is a  $C^*$ -algebra without unit and  $B$  is a Čebyšev  $C^*$ -subalgebra of  $A$ , then  $A = B$  (compare [29, Theorem 4]). The first main difference in the setting of JBW\*-triples is the existence of Čebyšev JBW\*-subtriples with arbitrary dimensions; complex Hilbert spaces and spin factors give a complete list of examples (compare Remark 7 and comments before it).

In our main result about Čebyšev JBW\*-subtriples (cf. Theorem 14), we establish the following criterium: Let  $N$  be a non-zero Čebyšev JBW\*-subtriple of a JBW\*-triple  $M$ . Then exactly one of the following statements holds:

- (a)  $N$  is a rank one JBW\*-triple with  $\dim(N) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $N$  may be a closed subspace of arbitrary dimension and  $M$  may have arbitrary rank;
- (b)  $N = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $M$ ;
- (c)  $N$  and  $M$  have rank two, but  $N$  may have arbitrary dimension;
- (d)  $N$  has rank greater or equal than three and  $N = M$ .

We provide examples of infinite dimensional proper Čebyšev JBW\*-subtriples of JBW\*-triples (see Remark 7). We apply the solution of the minimum

covering sphere problem in the Euclidean space  $\ell_2^m$  to present new examples of Čebyšev subspaces of classical Banach spaces (cf. Remark 12), and to construct an example of a rank-one Hilbert space which is a Čebyšev JBW\*-subtriple of a rank- $n$  JBW\*-triple, where  $n$  is an arbitrary natural number (cf. Remark 13).

It should be remarked at this point that the techniques applied by Robertson, Yost [30, 31] and Pedersen [29] in the setting of von Neumann algebras do not make any sense in the wider setting of JBW\*-triples. The techniques developed in this paper are completely independent and provide new arguments to understand the Čebyšev von Neumann subalgebras of a von Neumann algebra (Corollary 15).

## 2. ONE-DIMENSIONAL ČEBYŠEV SUBSPACES AND SUBTRIPLES OF JBW\*-TRIPLES

A complex Jordan triple system is a complex linear space  $E$  equipped with a triple product which is bilinear and symmetric in the external variables and conjugate linear in the middle one and satisfies the Jordan identity:

$$(2.1) \quad L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all  $x, y, a, b, c \in E$ , where  $L(x, y) : E \rightarrow E$  is the linear mapping given by  $L(x, y)z = \{x, y, z\}$ .

A *JB\*-triple* is a complex Jordan triple system  $E$  which is a Banach space satisfying the additional “*geometric*” axioms:

- (a) For each  $x \in E$ , the operator  $L(x, x)$  is hermitian with non-negative spectrum;
- (b)  $\|\{x, x, x\}\| = \|x\|^3$  for all  $x \in E$ .

Every C\*-algebra is a JB\*-triple with respect to the triple product given by

$$(2.2) \quad \{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).$$

Every JB\*-algebra (i.e. a complex Jordan Banach \*-algebra satisfying

$$\|U_a(a^*)\| = \|a\|^3,$$

for every element  $a$ , where  $U_a(x) := 2(a \circ x) \circ a - a^2 \circ x$ , cf. [20, §3.8]) is a JB\*-triple under the triple product defined

$$(2.3) \quad \{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

The space  $B(H, K)$  of all bounded linear operators between complex Hilbert spaces, although rarely is a C\*-algebra, is a JB\*-triple with the product defined in (2.2). In particular, every complex Hilbert space is a JB\*-triple.

Other examples of JB\*-triples are given by the so-called *Cartan factors*. A Cartan factor of type 1 is a JB\*-triple which coincides with the Banach space  $B(H, K)$  of bounded linear operators between two complex Hilbert spaces,  $H$  and  $K$ , where the triple product is defined by (2.2). Cartan factors of types 2

and 3 are JB\*-triples which can be identified the subtriples of  $B(H)$  defined by  $II^{\mathbb{C}} = \{x \in B(H) : x = -jx^*j\}$  and  $III^{\mathbb{C}} = \{x \in B(H) : x = jx^*j\}$ , respectively, where  $j$  is a conjugation on  $H$ . A Cartan factor of type 4 or IV is a spin factor, that is, a complex Hilbert space provided with a conjugation  $x \mapsto \bar{x}$ , where the triple product and the norm are defined by

$$\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

and  $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - |\langle x/\bar{x} \rangle|^2}$ , respectively. The Cartan factors of types 5 and 6 consist of finite dimensional spaces of matrices over the eight dimensional complex Cayley division algebra  $\mathbb{O}$ ; the type VI is the space of all hermitian 3x3 matrices over  $\mathbb{O}$ , while the type V is the subtriple of 1x2 matrices with entries in  $\mathbb{O}$  (compare [27], [18], and [12, §2.5]).

A JB\*-triple  $W$  is called a *JBW\*-triple* if it has a predual  $W_*$ . It is known that a JBW\*-triple admits a unique isometric predual and its triple product is separately  $\sigma(W, W_*)$ -continuous (see [3]). The second dual  $E^{**}$  of a JB\*-triple  $E$  is a JBW\*-triple with respect to a triple product which extends the triple product of  $E$  (cf. [14]).

For more detail of the properties of JB\*-triples and JBW\*-triples the reader is referred to the monographs [12] and [11].

Given an element  $a$  in a JB\*-triple  $E$ , the symbol  $Q(a)$  will denote the conjugate linear operator on  $E$  defined by  $Q(a)(x) = \{a, x, a\}$ .

An element  $e \in E$  is called a *tripotent* when  $\{e, e, e\} = e$ . Each tripotent  $e \in E$  induces a decomposition of  $E$ , called *the Peirce decomposition*, in the form  $E = E_2(e) \oplus E_1(e) \oplus E_0(e)$ , where  $E_i(e)$  is the  $\frac{i}{2}$  eigenspace of the operator  $L(e, e)$ ,  $i = 0, 1, 2$ . This decomposition satisfies the following *Peirce rules*:

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0$$

and

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

when  $i - j + k \in \{0, 1, 2\}$  and is zero otherwise. The projection  $P_k(e)$  of  $E$  onto  $E_k(e)$  is called the *Peirce  $k$ -projection*. It is known that Peirce projections are contractive (cf. [17, Corollary 1.2]) and satisfy:

$$P_2(e) = Q(e)^2, \quad P_1(e) = 2(L(e, e) - Q(e)^2),$$

and

$$P_0(e) = Id_E - 2L(e, e) + Q(e)^2.$$

The separate weak\*-continuity of the triple product of a JBW\*-triple  $M$  implies that Peirce projections associated with a tripotent  $e$  in  $M$  are weak\*-continuous.

It is known that the Peirce-2 subspace  $E_2(e)$  is a JB\*-algebra with unit  $e$ , Jordan product  $x \circ_e y := \{x, e, y\}$  and involution  $x^{*e} := \{e, x, e\}$ , respectively. Since surjective linear isometries and triple isomorphisms on a JB\*-triple

coincide (cf. [24, Proposition 5.5]), the triple product in  $E_2(e)$  is uniquely given by

$$\{x, y, z\} = (x \circ_e y^{*e}) \circ_e z + (z \circ_e y^{*e}) \circ_e x - (x \circ_e z) \circ_e y^{*e},$$

$x, y, z \in E_2(e)$ .

We shall make use of the following property: given a tripotent  $e \in E$  and an element  $\lambda$  in the unit sphere of  $\mathbb{C}$ , the mapping:

$$(2.4) \quad S_\lambda(e) : E \rightarrow E, \quad S_\lambda(e) = \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e),$$

is a surjective linear isometry on  $E$  and a triple isomorphism (compare [17, Lemma 1.1]).

A tripotent  $e \in E$  is said to be *unitary* if the operator  $L(e, e)$  coincides with the identity map  $I_E$  on  $E$ ; that is,  $E_2(e) = E$ . We shall say that  $e$  is *complete* or *maximal* when  $E_0(e) = E$ . When  $E_2(e) = P_2(e)(E) = \mathbb{C}e \neq \{0\}$ , we say that  $e$  is *minimal*.

The complete tripotents of a JB\*-triple  $E$  coincide with the real and complex extreme points of its closed unit ball  $E_1$  (cf. [5, Lemma 4.1] and [25, Proposition 3.5] or [12, Theorem 3.2.3]). Consequently, the Krein-Milman theorem assures that every JBW\*-triple admits an abundant set of complete tripotents [12, Corollary 3.2.4].

When  $a$  is an element in a JBW\*-triple  $M$ , the sequence  $(a^{\frac{1}{2^n-1}})$  converges in the weak\*-topology of  $M$  to a tripotent, denoted by  $r(a)$ , called the *range tripotent of  $a$* . The tripotent  $r(a)$  is the smallest tripotent  $e \in M$  satisfying that  $a$  is positive in the JBW\*-algebra  $M_2(e)$  (see [15, page 322]).

Let  $a$  be an element in a JB\*-triple  $E$ . It is known that the JB\*-subtriple  $E_a$  generated by  $a$ , identifies with some  $C_0(L)$  where  $\|a\| \in L \subseteq [0, \|a\|]$  with  $L \cup \{0\}$  compact (cf. [24, 1.15]). Moreover, there exists a triple isomorphism  $\Psi : E_a \rightarrow C_0(L)$  such that  $\Psi(a)(t) = t$ . Clearly, the range tripotent  $r(a)$  can be identified with the characteristic function  $\chi_{(0, \|a\|] \cap L} \in C_0(L)^{**}$  (see [7, beginning of §2]).

We recall that an element  $x$  in a Jordan algebra  $\mathcal{J}$  with unit  $e$  is called *invertible* if there exists an element  $y$  such that  $x \circ y = e$  and  $x^2 \circ y = x$ . The element  $y$  is called *the inverse of  $x$* , and is denoted by  $x^{-1}$ . Inverse of any element  $x$  in a Jordan algebra  $\mathcal{J}$  is unique whenever it exists. The set of all invertible elements in  $\mathcal{J}$  is denoted by  $\mathcal{J}^{-1}$ .

An element  $a$  in a JB\*-triple  $E$  is called *von Neumann regular* if and only if there exists  $b \in E$  such that

$$Q(a)(b) = a, \quad Q(b)(a) = b, \quad \text{and} \quad [Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0.$$

When  $a$  is von Neumann regular, the (unique) element  $b \in E$  satisfying the above conditions is called *the generalized inverse of  $a$* , and is denoted by  $a^\dagger$ . It is known that an element  $a \in E$  is von Neumann regular if, and only if,  $Q(a)$  has norm-closed image if, and only if, the range tripotent  $r(a)$  of  $a$  lies in  $E$  and  $a$  is positive and invertible element of the JB\*-algebra

$E_2(r(a))$  (compare [10]). Furthermore, when  $a$  is von Neumann regular,  $Q(a)Q(a^\dagger) = Q(a^\dagger)Q(a) = P_2(r(a))$  and  $L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$  [10, page 192].

Given a pair of elements  $a, b$  in a JB\*-triple  $E$ , the *Bergmann operator* associated to  $a$  and  $b$  is the mapping  $B(a, b) : E \rightarrow L(E)$  defined by  $B(a, b) = Id_E - 2L(a, b) + Q(a)Q(b)$  (cf. [12, page 22]).

An element  $a$  in a JB\*-triple  $E$  is said to be *Brown-Pedersen quasi-invertible* (*BP-quasi-invertible* for short) when it is von Neumann regular with generalized inverse  $b$  such that the Bergman operator  $B(a, b)$  vanishes; in such a case,  $b$  is called *the BP-quasi inverse* of  $a$ . The set of BP-quasi invertible elements in  $E$  is denoted by  $E_q^{-1}$  [35]. It is established in [35] that an element  $a \in E$  is BP-quasi-invertible if, and only if, one of the following equivalent statements holds:

- (i)  $a$  is von Neumann regular, and its range tripotent  $r(a)$  is an extreme point of the closed unit ball  $E_1$  of  $E$  (i.e.  $r(a)$  is a complete tripotent of  $E$ );
- (ii) There exists a complete tripotent  $e \in E$  such that  $a$  is positive and invertible in the JB\*-algebras  $E_2(e)$ .

We recall that two elements  $a, b$  in a JB\*-triple  $E$ , are said to be *orthogonal* (written  $a \perp b$ ) if  $L(a, b) = 0$ . Lemma 1 in [8] shows that  $a \perp b$  if and only if one of the following nine statements holds:

$$\begin{aligned}
 (2.5) \quad & \{a, a, b\} = 0; & a \perp r(b); & r(a) \perp r(b); \\
 & E_2^{**}(r(a)) \perp E_2^{**}(r(b)); & r(a) \in E_0^{**}(r(b)); & a \in E_0^{**}(r(b)); \\
 & b \in E_0^{**}(r(a)); & E_a \perp E_b & \{b, b, a\} = 0.
 \end{aligned}$$

Let  $e$  be a tripotent in a JB\*-triple  $E$ . Lemma 1.3(a) in [17] shows that

$$\|x_2 + x_0\| = \max\{\|x_2\|, \|x_0\|\},$$

for every  $x_2 \in E_2(e)$  and every  $x_0 \in E_0(e)$ . Combining this result with the equivalences in (2.5) we see that

$$(2.6) \quad \|a + b\| = \max\{\|a\|, \|b\|\},$$

whenever  $a$  and  $b$  are orthogonal elements in a JB\*-triple.

Given a subset  $M \subseteq E$ , we write  $M_E^\perp$  (or simply  $M^\perp$ ) for the (orthogonal) annihilator of  $M$  defined by  $M_E^\perp = \{y \in E : y \perp x, \forall x \in M\}$ . If  $e \in E$  is a tripotent, then  $\{e\}^\perp = E_0(e)$ , and  $\{a\}^\perp = (E^{**})_0(r(a)) \cap E$ , for every  $a \in E$  (cf. [9, Lemma 3.2]).

**Lemma 3.** *Let  $V$  be a non-zero Čebyšev subspace of a JBW\*-triple  $M$ . Then  $V \cap M_q^{-1} \neq \emptyset$ , where  $M_q^{-1}$  denotes the set of BP-quasi invertible elements of  $M$ .*



*Proof.* Arguing by contradiction, we suppose that  $V \cap M_q^{-1} = \emptyset$ .

Let us take  $x \in V$  with  $\|x\| = 1$ . By assumptions,  $x \notin M_q^{-1}$ . Under these conditions, the range complete tripotent of  $x$ ,  $r(x)$  is not complete in  $M$  or  $x$  is not invertible in the JBW\*-algebra  $M_2(r(x))$ . By [22, Lemma 3.12], there exists a complete tripotent  $e$  in  $M$  such that  $r(x) \leq e$ .

We shall identify the JB\*-subtriple,  $M_x$ , of  $M$  generated by  $x$  with some  $C_0(L)$  where  $1 = \|x\| \in L \subseteq [0, \|1\|]$  with  $L \cup \{0\}$  compact (cf. [24, 1.15]). We further know that there exists a triple isomorphism  $\Psi : M_x \rightarrow C_0(L)$  such that  $\Psi(x)(t) = t$ , and the range tripotent  $r(x)$  identifies with the characteristic function  $\chi_{(0, \|x\|] \cap L} \in C_0(L)^{**}$  (see page 2). It is clear that, under this identification,

$$\|r(x) - \lambda x\| = 1 - |\lambda| \inf\{|x(t)| : t \in L\} \leq 1,$$

for every  $|\lambda| \leq 1$  in  $\mathbb{C}$ . When  $e = r(x)$ , the element  $x$  is not invertible in the JBW\*-algebra  $M_2(r(x))$ , and hence  $\|e - x\| = \|r(x) - x\| = 1$ . When  $e \not\geq r(x)$ , we have  $\|e - r(x)\| = 1$ . Thus, applying  $e - r(x) \perp r(x)$  and (2.6), we further know that

$$\|e - \lambda x\| = \|e - r(x) + r(x) - \lambda x\| = \max\{\|e - r(x)\|, \|r(x) - \lambda x\|\} = 1.$$

We observe that, since  $e$  is a complete tripotent,  $e \in M_q^{-1}$ , and hence  $e \notin V$ . Since  $V$  is a Čebyšëv subspace, there exists a unique best approximation,  $c_V(e) \in V$ , of  $e$  in  $V$  satisfying  $\text{dist}(e, V) = \|e - c_V(e)\| > 0$ .

If  $\text{dist}(e, V) = \|e - c_V(e)\| \geq 1$ , we would have  $1 = \|e\| \geq \text{dist}(e, V) = 1$ , and

$$1 = \|e - c_V(e)\| = \text{dist}(e, V) = \|e - \lambda x\|,$$

for every  $|\lambda| \leq 1$ , contradicting the uniqueness of the best approximation of  $e$  in  $V$ . We can therefore assume that  $\text{dist}(e, V) < 1$ . Consequently, there exists  $y \in V$  with  $\|e - y\| < 1$ . Corollary 2.4. in [23] implies that  $y \in M_q^{-1} \cap V$ , which is impossible.  $\square$

Let  $e$  be a tripotent in a JB\*-triple  $E$ . Let us recall that  $e$  is a tripotent in the JBW\*-triple  $E^{**}$ , and that Peirce projections associated with  $e$  on  $E^{**}$  are weak\*-continuous. Goldstine's theorem assures that  $E$  is weak\*-dense in  $E^{**}$ , and hence,  $E_k^{**}(e)$  coincides with the weak\*-closure of  $E_k(e)$  in  $E^{**}$ , for every  $k = 0, 1, 2$ . In particular,  $e$  is complete in  $E^{**}$  whenever  $e$  is a complete tripotent in  $E$ . Moreover, since the orthogonal complement of a tripotent  $e$  in a JB\*-triple  $F$  coincides with  $F_0(e)$ , we have:

**Lemma 4.** *Let  $e$  be a complete tripotent in a JB\*-triple  $E$ . Then  $\{e\}_{E^{**}}^\perp = \{0\}$ , that is,  $e$  is not orthogonal to any non-zero element in  $E^{**}$ .  $\square$*

The following technical result is part of the folklore in the theory of best approximation (see [30, Lemma 3] or [33, Theorem 2.1]).

**Lemma 5.** ([30, Lemma 3]). *Let  $x$  be an element in complex a Banach space  $X$  such that  $\mathbb{C}x$  is not a Čebyšëv subspace of  $X$ . Then there exists an*



extreme point  $\phi$  of the closed unit ball of  $X^*$ , a vector  $y \in X$  and a scalar  $\lambda \in \mathbb{C} \setminus \{0\}$  such that

- (a)  $\phi(x) = 0$ ;
- (b)  $\phi(y) = \|y\| = \|y - \lambda x\|$ .

□

We can characterize now the one dimensional Čebyšev subspaces of a JBW\*-triple.

**Theorem 6.** *Let  $x$  be a non-zero element in a JBW\*-triple  $M$ . The following statements are equivalent:*

- (a)  $\mathbb{C}x$  is a Čebyšev subspace of  $M$ ;
- (b)  $x$  is a Brown-Pedersen quasi-invertible element in  $M$ ;

*Proof.* The implication (a)  $\Rightarrow$  (b) follows from Lemmas 3.

(b)  $\Rightarrow$  (a) Suppose  $x$  is BP-quasi invertible in  $M$ . We note that the support tripotent,  $r(x)$ , of  $x$  is complete in  $M$ , and hence a complete tripotent in  $M^{**}$  (cf. Lemma 4 and comments before it).

Suppose that  $\mathbb{C}x$  is not a Čebyšev subspace of  $M$ . By Lemma 5 there exists an extreme point  $\phi$  of the closed unit ball of  $M^*$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$ , and  $y \in M$  such that  $\phi(x) = 0$  and  $\phi(y) = \|y\| = \|y - \lambda x\|$ .

The support tripotent  $v = s(\phi)$  of  $\phi$  in  $M^{**}$  is a (non-zero) minimal tripotent in  $M^{**}$  satisfying  $\phi = P_2(v)^*\phi = \phi P_2(v)$  and  $\phi(z)v = P_2(v)(z)$ ,  $\forall z \in M^{**}$  (cf. [17, Proposition 4]). Therefore,  $P_2(v)(x) = \phi(x)v = 0$ .

We may suppose that  $\|y\| = 1$ . Since  $P_2(v)(y) = \phi(y)v = v$ , Lemma 1.6 in [17] implies that  $P_1(v)(y) = 0$ , which shows that  $y = v + P_0(v)y$ . We similarly get  $P_1(v)(y - \lambda x) = 0$  (we simply observe that  $\phi(y - \lambda x) = \|y\| = \|y - \lambda x\| = 1$ ). Therefore,  $P_1(v)(x) = 0$ , and  $x = P_0(v)x \in (M^{**})_0(v) = ((M^{**})_2(v))^\perp$ , implying that  $x \perp v$ . The equivalent statements in (2.5) prove that  $r(x) \perp v$ , which contradicts Lemma 4. □

The above Theorem 6 generalizes the previously commented results obtained by Robertson [30] (compare Theorem 1). In order to find a triple version of the reformulation established by Pedersen in [29, Theorem 2], stated as statement (c) in page 2, we recall some notation.

For each functional  $\varphi$  in the predual of a JBW\*-triple  $W$ , and for each  $z$  in  $W$  with  $\varphi(z) = \|\varphi\|$ , and  $\|z\| = 1$ , the mapping  $x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{1/2}$  defines a pre-Hilbertian semi-norm on  $W$ . Moreover,  $\varphi\{x, x, w\} = \varphi\{x, x, z\}$  whenever  $w \in W$  with  $\varphi(w) = \|\varphi\|$  and  $\|w\| = 1$  (cf. [1, Proposition 1.2]). It is known that

$$(2.7) \quad |\varphi(x)| \leq \|x\|_\varphi,$$

for every  $x \in W$  (see [2, page 258]).

The inequality in (2.7) together with Lemma 5 imply the following property: Let  $x$  be a non-zero element in a JBW\*-triple  $M$  such that  $\mathbb{C}x$  is a Čebyšev subspace of  $M$ . Then for each extreme point  $\varphi$  of the closed unit

ball of  $M^*$  we have  $\|x\|_\varphi \geq 0$ . It would be interesting to know under what additional hypothesis, the condition  $\|x\|_\varphi \geq 0$ , for every extreme point  $\varphi$  of the closed unit ball of  $M^*$ , implies that  $x$  is BP-quasi invertible.

### 3. ČEBYŠEV SUBTRIPLES OF $\text{JBW}^*$ -TRIPLES

In this section, we shall determine the  $\text{JBW}^*$ -subtriples of a  $\text{JBW}^*$ -triple  $M$  which are Čebyšev subspaces in  $M$ . Let us recall that in the case of an infinite dimensional von Neumann algebra  $M$ , if a finite dimensional von Neumann subalgebra  $N$  of  $M$  is a Čebyšev subspace in  $M$  then  $N$  must be one dimensional (compare Theorem 2 or [30, Theorem 6]). Furthermore, an infinite dimensional  $C^*$ -algebra  $A$  admits a finite dimensional  $*$ -subalgebra  $B$  which is also a Čebyšev in  $A$  if and only if  $A$  is unital and  $B = \mathbb{C}1$  (cf. [31, Corollary 1.4]). The scarcity of non-trivial Čebyšev  $C^*$ -subalgebras in general  $C^*$ -algebras can be better understood with the following result due to G.K. Pedersen: If  $A$  is a  $C^*$ -algebra without unit and  $B$  is a Čebyšev  $C^*$ -subalgebra of  $A$ , then  $A = B$  (compare [29, Theorem 4]).

The first main difference in the setting of  $\text{JB}^*$ -triples is the existence of Čebyšev  $\text{JB}^*$ -subtriples with arbitrary dimensions. For example, let  $E = H$  be a complex Hilbert space regarded as a type 1 Cartan factor with the Hilbert norm and the product

$$(3.1) \quad \{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x),$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $H$ . It is known that elements in the unit sphere of a complex Hilbert  $H$  space regarded as a type 1 Cartan factor are precisely the complete tripotents of  $H$ . The *Orthogonal Projection theorem* tells that any closed subspace of  $H$  is a Čebyšev subspace of  $H$  and clearly a  $\text{JB}^*$ -subtriple.

The following remark provides an additional example.

*Remark 7.* Let  $E$  be a spin factor with triple product and norm given by

$$\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

and  $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - |\langle x/\bar{x} \rangle|^2}$ , respectively, where  $x \mapsto \bar{x}$  is a conjugation on  $E$ , and  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $E$ . Let  $K$  be a closed subspace of  $E$  with  $\bar{K} = K$ . Clearly,  $K$  is a  $\text{JB}^*$ -subtriple of  $E$ . Since  $K$  is a closed subspace of the complex Hilbert space  $E$ , there exists an orthogonal projection  $P$  of  $E$  onto  $K$ . Since  $E = K \oplus H$ , where  $H = (I - P)(E)$  with  $\langle K/H \rangle = 0$ . Since  $\bar{K} = K$ , we also have  $\bar{H} = H$ . Given  $\eta \in K$  and  $\xi \in H$ , it is easy to check that

$$\begin{aligned} \|\eta + \xi\|^2 &= \langle \eta + \xi/\eta + \xi \rangle + \sqrt{\langle \eta + \xi/\eta + \xi \rangle^2 - |\langle \eta + \xi/\bar{\eta} + \bar{\xi} \rangle|^2} \\ &= \langle \eta/\eta \rangle + \langle \xi/\xi \rangle + \sqrt{\langle \eta/\eta \rangle^2 - |\langle \eta/\bar{\eta} \rangle|^2 + \langle \xi/\xi \rangle^2 - |\langle \xi/\bar{\xi} \rangle|^2} \\ &\geq \langle \eta/\eta \rangle + \sqrt{\langle \eta/\eta \rangle^2 - |\langle \eta/\bar{\eta} \rangle|^2} = \|\eta\|^2. \end{aligned}$$

Moreover,  $\|\eta + \xi\| = \|\eta\|$  if and only if  $\xi = 0$ . This shows that  $P : E \rightarrow E$  is a bi-contractive for the norm  $\|\cdot\|$ , and for each  $x \in E$ ,  $P(x)$  is the unique best approximation of  $x$  in  $K$ . Therefore,  $K$  is a Čebyšev JB\*-subtriple of  $E$ . We observe that the dimensions of  $E$  and  $K$  can be arbitrarily big.

We can present now our conclusions on Čebyšev JB\*-subtriples.

The next property of Čebyšev subspaces is probably part of the folklore in the theory of best approximation in normed spaces, but we couldn't find an exact reference.

**Lemma 8.** *Let  $V$  be a Čebyšev subspace of a normed space  $X$ . For each  $x \in X$ , we denote by  $c_V(x)$  the unique element in  $V$  satisfying  $\|x - c_V(x)\| = \text{dist}(x, V)$ . Let  $P : X \rightarrow X$  be a contractive projection such that  $P(V) \subseteq V$ . Then*

$$P(c_V(P(x))) = c_V(P(x)),$$

*for every  $x \in X$ . Furthermore,  $P(V)$  is a Čebyšev subspace of the normed space  $P(X)$ , and for each  $x \in X$ ,  $c_{P(V)}(P(x)) = P(c_V(x))$ .*

*Proof.* Let  $x$  be an element in  $X$ . The condition  $\|P\| \leq 1$  implies that

$$\|P(x) - P(c_V(P(x)))\| \leq \|P(x) - c_V(P(x))\| = \text{dist}(P(x), V).$$

The element  $P(c_V(P(x))) \in P(V) \subseteq V$ . Thus, the uniqueness of the best approximation in  $V$  proves that  $P(c_V(P(x))) = c_V(P(x))$ . The rest is clear.  $\square$

**Proposition 9.** *Let  $F$  be a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ . Suppose  $e$  is a non-zero tripotent in  $F$ . Then  $E_0(e) = F_0(e)$ . Consequently, every complete tripotent in  $F$  is complete in  $E$ .*

*Proof.* Since  $e$  is a tripotent in  $F$  and the latter is a JB\*-subtriple of  $E$ ,  $e$  is a tripotent in  $E$  and  $F_0(e) \subseteq E_0(e)$ . Arguing by contradiction, let us assume that there exists  $b \in E_0(e) \setminus F_0(e) = E_0(e) \setminus F \neq \emptyset$ . Since  $\text{dist}(b, F) > 0$  and  $F$  is a Čebyšev subspace, there exists a unique  $c_F(b) \in F$  such that  $\|b - c_F(b)\| = \text{dist}(b, F)$ .

Since  $P_0(e)(F) \subseteq F$  and  $P_0(e)(b) = b$ , Lemma 8 implies that

$$P_0(e)(c_F(b)) = c_F(b) \in F_0(e).$$

Having in mind that  $e \in E_2(e) \perp E_0(e) \ni b - c_F(b)$ , we deduce, via (2.6), that

$$\|b - c_F(b) - \lambda e\| = \max\{\|b - c_F(b)\|, |\lambda|\} = \|b - c_F(b)\| = \text{dist}(b, F),$$

for every  $|\lambda| \leq \text{dist}(b, F)$ . This contradicts the uniqueness of the best approximation,  $c_F(b)$ , of  $b$  in  $F$ , because  $c_F(b) + \lambda e \in F$  for every  $|\lambda| \leq \text{dist}(b, F)$ .  $\square$

**Proposition 10.** *Let  $F$  be a Čebyšëv  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Suppose  $e$  is a tripotent in  $F$  with  $F_0(e) = \{e\}_F^\perp \neq 0$ . Then  $E_2(e) = F_2(e)$ .*

*Proof.* Clearly  $F_2(e) \subseteq E_2(e)$ . We have to show that  $E_2(e) \subseteq F_2(e)$ . Suppose, on the contrary, that  $E_2(e) \setminus F_2(e) = E_2(e) \setminus F \neq \emptyset$ . Pick  $b \in E_2(e) \setminus F$ . Since  $F$  is a Čebyšëv subspace of  $E$ , there exists a unique  $c_F(b) \in F$  satisfying  $\|b - c_F(b)\| = \text{dist}(b, F) > 0$ .

By Lemma 8 applied to  $P = P_2(e)$ ,  $X = E$  and  $V = F$ , we deduce that  $P_2(e)(c_F(b)) = c_F(b)$ .

By hypothesis,  $F_0(e) = \{e\}_F^\perp \neq 0$ . So, there exists a norm-one element  $z \in F_0(e)$ . The conditions  $b \in E_2(e)$ ,  $c_F(b) \in F_2(e)$  and  $z \in F_0(e)$  combined with 2.6 give

$$\|b - c_F(b) - \lambda z\| = \max\{\|b - c_F(b)\|, |\lambda|\} = \|b - c_F(b)\| = \text{dist}(b, F),$$

for every  $|\lambda| \leq \text{dist}(b, F)$ , which contradicts the uniqueness of the best approximation of  $b$  in  $F$  because  $c_F(b) - \lambda z \in F$ , for every  $\lambda$  in the above conditions.  $\square$

Let  $e$  and  $v$  be tripotents in a  $JB^*$ -triple  $E$ . We shall say that  $v \leq e$ , when  $e - v$  is a tripotent in  $E$  with  $e - v \perp v$  (compare the notation in [17]).

Let  $E$  be a  $JB^*$ -triple. A subset  $S \subseteq E$  is said to be *orthogonal* if  $0 \notin S$  and  $x \perp y$  for every  $x \neq y$  in  $S$ . The minimal cardinal number  $r$  satisfying  $\text{card}(S) \leq r$  for every orthogonal subset  $S \subseteq E$  is called the *rank* of  $E$  (and will be denoted by  $r(E)$ ). Given a tripotent  $e \in E$ , the rank of the Peirce-2 subspace  $E_2(e)$  will be called the rank of  $e$ .

Theorem 3.1 in [4] combined with Proposition 4.5.(iii) in [6] assure that a  $JB^*$ -triple is reflexive if and only if it is isomorphic to a Hilbert space if, and only if, it has finite rank.

Suppose  $E$  is a rank-one  $JB^*$ -triple. The above comments show that  $E$  is reflexive and hence a  $JBW^*$ -triple. Let  $e$  be a complete tripotent in  $E$ . Since the rank of  $e$  is smaller than the rank of  $E$ , we deduce that  $e$  is a minimal tripotent in  $E$ . Proposition 3.7 in [9] and its proof show that  $E = \{e\}^{\perp\perp} = \{0\}^\perp$  is a rank-one Cartan factor of the form  $L(H, \mathbb{C})$ , where  $H$  is a complex Hilbert space or a type 2 Cartan factor  $II_3$  (it is known that  $II_3$  is  $JB^*$ -triple isomorphic to a 3-dimensional complex Hilbert space). We have proved the following:

**Lemma 11.** *Every  $JB^*$ -triple of rank one is  $JB^*$ -isomorphic (and hence isometric) to a complex Hilbert space regarded as a type 1 Cartan factor.  $\square$*

The above result is also stated in [13, Corollary in page 308].

We have already commented that orthogonal elements are  $M$ -orthogonal in the sense of the geometric theory of Banach spaces (see (2.6)). We shall state next another results of geometric nature. Let  $u$  and  $v$  be two non-zero tripotents in a  $JB^*$ -triple  $E$ . We recall that  $u$  and  $v$  are *colinear* (written  $u \top v$ ) when  $u \in E_1(v)$  and  $v \in E_1(u)$  (cf. [13, page 296]). Suppose  $u \top v$  in  $E$ .

Clearly, the JB\*-subtriple  $E_{u,v}$  of  $E$  generated by  $u$  and  $v$  is algebraically isomorphic to  $\mathbb{C}u \otimes \mathbb{C}v$ . We observe that  $u$  and  $v$  are minimal colinear tripotents in  $E_{u,v}$ . It follows from [17, Proposition 5] that  $E_{u,v}$  is JB\*-triple isomorphic and hence isometric to  $M_{1,2}(\mathbb{C})$  (regarded as a type 1 Cartan factor). We consequently have

$$(3.2) \quad \|\lambda u + \mu v\| = (|\lambda|^2 + |\mu|^2)^{\frac{1}{2}},$$

for every  $\lambda, \mu \in \mathbb{C}$ . It should be also noted here that, in a Hilbert space  $F$  regarded as a type 1 Cartan factor with product given in (3.1). In this case, the tripotents of  $F$  are precisely the elements in its unit sphere, and the relation of being Hilbert-orthogonal is exactly the relation of colinearity in terms of the triple product.

We have shown several examples of Hilbert spaces (regarded as a type 1 Cartan factor) which are Čebyšev JB\*-subtriples of JB\*-triples of rank one and two. We present next more examples of Hilbert spaces which are Čebyšev JB\*-subtriples of JB\*-triples having a bigger rank. The first example is a construction with classical Banach spaces and the second one is an isometric translation to the setting of JB\*-triples.

*Remark 12.* Let  $H$  be complex Hilbert space of dimension 2 with norm

denoted by  $\|\cdot\|_2$ . We consider the Banach space  $X = \overbrace{H \oplus^{\ell_\infty} \dots \oplus^{\ell_\infty} H}^{(n)}$  ( $n \geq 2$ ). Let  $\{\xi_1, \xi_2\}$  be an orthonormal basis of  $H$ . Each  $h \in H$  writes uniquely in the form  $h = \lambda_1 \xi_1 + \lambda_2 \xi_2$ . Let  $V$  denote the 2-dimensional subspace of  $X$  generated by the vectors  $e_1 = (\xi_1, \dots, \xi_1)$  and  $e_2 = (\xi_2, \dots, \xi_2)$ . That is, every vector in  $V$  writes in the form  $\lambda e_1 + \mu e_2$ . Clearly,

$$\begin{aligned} \|\lambda e_1 + \mu e_2\| &= \|\lambda(\xi_1, \dots, \xi_1) + \mu(\xi_2, \dots, \xi_2)\|_2 \\ &= \max_{i=1, \dots, n} \|\lambda \xi_1 + \mu \xi_2\|_2 = \sqrt{|\lambda|^2 + |\mu|^2}, \end{aligned}$$

and hence  $V$  is isometrically isomorphic to a Hilbert space.

We claim that  $V$  is a Čebyšev subspace of  $X$ . Indeed, let  $x = (h_1, \dots, h_n)$  be an element in  $X$  and let  $\lambda e_1 + \mu e_2 \in V$ . We write  $h_i = \lambda_1^i \xi_1 + \lambda_2^i \xi_2$ . We write the formula for the distance from  $x$  to  $V$  in the form:

$$\begin{aligned} \text{dist}(x, V)^2 &= \inf_{\lambda, \mu \in \mathbb{C}} \|(h_1, \dots, h_n) - \lambda e_1 - \mu e_2\|^2 \\ &= \inf_{\lambda, \mu \in \mathbb{C}} \max_{i=1, \dots, n} \|\lambda_1^i \xi_1 + \lambda_2^i \xi_2 - \lambda \xi_1 - \mu \xi_2\|_2^2 \\ &= \inf_{\lambda, \mu \in \mathbb{C}} \max_{i=1, \dots, n} (|\lambda_1^i - \lambda|^2 + |\lambda_2^i - \mu|^2)^{\frac{1}{2}} = \inf_{\lambda, \mu \in \mathbb{C}} \max_{i=1, \dots, n} \text{dist}_{\mathbb{C}^2}((\lambda_1^i, \lambda_2^i), (\lambda, \mu)). \end{aligned}$$

Our problem is equivalent to determine a point  $(\lambda, \mu) \in \mathbb{C}^2$  so that the maximum Euclidean distance from  $(\lambda, \mu)$  to the points  $(\lambda_1^i, \lambda_2^i) \in \mathbb{C}^2$

( $i = 1, \dots, n$ ) is minimized, where  $\mathbb{C}^2$  is equipped with the Euclidean distance  $\|(\lambda, \mu)\|_2 = \sqrt{|\lambda|^2 + |\mu|^2}$ . This problem is commonly called “the Euclidean delivery problem” or “the min-max location problem” or “the minimum covering sphere problem”. It is known that an equivalent reformulation of the problem is:

$$\text{Min}\{\rho : (\lambda, \mu) \in \mathbb{C}^2, \rho > 0, \|(\lambda_1^i, \lambda_2^i) - (\lambda, \mu)\|_2 \leq \rho, \forall i\}.$$

The goal is to find the circle of center  $(\lambda, \mu) \in \mathbb{C}^2$  of smallest radius  $\rho$  that encloses all the points  $(\lambda_1^i, \lambda_2^i) \in \mathbb{C}^2$  ( $i = 1, \dots, n$ ).

It is well known that a solution to the the minimum covering sphere problem always exists, the center  $(\lambda, \mu)$  and the radius  $\rho$  are unique (cf. [21], [16]). This shows that every element  $x = (\lambda_1^1 \xi_1 + \lambda_2^1 \xi_2, \dots, \lambda_1^n \xi_1 + \lambda_2^n \xi_2)$  in  $X$  admits a unique best approximation in  $V$ , which proves the claim.

*Remark 13.* Let  $e$  and  $u$  be two colinear complete tripotents in a JB\*-triple  $E$ . Let us assume that we can find two sets  $\{e_1, \dots, e_n\}$  and  $\{u_1, \dots, u_n\}$  of mutually orthogonal tripotents in  $E_2(e)$  and  $E_2(u)$ , respectively, such that  $e_i \top u_i$ , for all  $i$ , and  $u_i \perp e_j$ , for every  $i \neq j$ . Take, for example,  $E = M_{n \times (2n)}(\mathbb{C})$ ,  $e = \sum_{i=1}^n w_{i,i}$ ,  $u = \sum_{i=1}^n w_{i,i+n}$ ,  $e_i = w_{i,i}$  and  $u_i = e = w_{i,i+n}$ , where  $w_{i,j}$  is the matrix with entry 1 at the position  $i, j$  and zero elsewhere.

Let  $F$  be the JB\*-subtriple of  $E$  generated by  $\{e_1, \dots, e_n, u_1, \dots, u_n\}$ , and let  $W$  be the closed JB\*-subtriple of  $F$  generated by  $\{e, u\}$ . For each  $i \in \{1, \dots, n\}$ ,  $e_i \top u_i$  and hence

$$\|\lambda_i e_i + \mu_i u_i\| = \sqrt{|\lambda_i|^2 + |\mu_i|^2},$$

that is, the subtriple,  $F_i$ , generated by  $e_i$  and  $u_i$  is a 2-dimensional complex Hilbert space (cf. (3.2)). Since, for each  $i \neq j$ ,  $\{e_i, u_i\} \perp \{e_j, u_j\}$  ( $F_i \perp F_j$ ), we deduce from (2.6) that  $\|x_i + x_j\| = \max\{\|x_i\|, \|x_j\|\}$ , for every  $x_i \in F_i, x_j \in F_j, i \neq j$ . Having in mind that  $F = F_1 \oplus^{\ell_\infty} \dots \oplus^{\ell_\infty} F_n$ , and  $F_i \equiv \ell_2^2$ , we can easily see that  $F$  is isometrically isomorphic to the space  $X$  in Remark 12. It is also easy to see that under the natural isometric identification of  $F$  and  $X$  in Remark 12, the JB\*-subtriple  $W$  is identified with the subspace  $V$  in that Remark. Therefore, it follows that  $W$  is a Čebyšev JB\*-subtriple of  $F$ . The JB\*-triple  $F$  has been constructed to have rank  $n$ .

The theorem describing the Čebyšev JBW\*-subtriples of a JBW\*-triple can be stated now. We shall show that the examples given in Remark 7 and the comments before it are essentially the unique examples of non-trivial Čebyšev JBW\*-subtriples.

**Theorem 14.** *Let  $N$  be a non-zero Čebyšev JBW\*-subtriple of a JBW\*-triple  $M$ . Then exactly one of the following statements holds:*

- (a)  *$N$  is a rank one JBW\*-triple with  $\dim(N) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $N$  may be a closed subspace of arbitrary dimension and  $M$  may have arbitrary rank;*

- (b)  $N = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $M$ ;
- (c)  $N$  and  $M$  have rank two, but  $N$  may have arbitrary dimension;
- (d)  $N$  has rank greater or equal than three and  $N = M$ .

*Proof.* We can always find a complete tripotent  $e$  in  $N$  (see the comments in page 6). Proposition 9 implies that  $e$  is complete in  $M$  (i.e.  $M_0(e) = \{0\}$ ). We have three possibilities:

- (i)  $e$  has rank one in  $N$ ;
- (ii)  $e$  has rank 2 in  $N$ ;
- (iii)  $e$  has rank greater or equal than 3 in  $N$ .

(i) Suppose first that  $e$  has rank one in  $N$ . In this case,  $e$  is a minimal and complete tripotent in  $N$ . Therefore,  $N$  is a complex Hilbert space regarded as a type 1 Cartan factor (cf. Lemma 11 or Proposition 3.7 in [9]).

The examples given before Remark 7 and in Remark 13 show that  $N$  may have arbitrary dimension and  $M$  may have rank as big as desired.

(ii) We assume now that  $e$  has rank 2 in  $N$ . Then there exist two non-zero minimal, mutually orthogonal tripotents  $e_1, e_2 \in N$  with  $e = e_1 + e_2$ . Propositions 9 and 10 show that  $M_2(e_j) = N_2(e_j)$ , and  $M_0(e_j) = N_0(e_j) \neq \{0\}$ , for every  $j$  in  $\{1, 2\}$ . Since  $M_2(e_j) = N_2(e_j) = \mathbb{C}e_j$ , we deduce that  $e_1$  and  $e_2$  are minimal tripotents in  $M$ . We also know that  $e = e_1 + e_2$  is a complete in  $M$  (i.e.  $M = M_2(e) \oplus M_1(e)$ ), which proves that  $M$  has rank two. The statement concerning the dimension of  $N$  follows from the example in Remark 7.

(iii) Suppose now that  $e$  has rank greater or equal than 3 in  $N$ . We shall show that  $M = N$ . Under the present assumptions, we can find three non-zero mutually orthogonal tripotents  $e_1, e_2, e_3$  with  $e_1 + e_2 + e_3 = e$ . Clearly,  $N_0(e_j + e_k) \neq \{0\}$ , for every  $k \neq j$  in  $\{1, 2, 3\}$ . Propositions 9 and 10 assure that  $M_2(e_j + e_k) = N_2(e_j + e_k)$ ,  $M_0(e_j + e_k) = N_0(e_j + e_k)$ ,  $M_2(e_j) = N_2(e_j)$ , and  $M_0(e_j) = N_0(e_j)$ , for every  $k \neq j$  in  $\{1, 2, 3\}$ . In the Peirce decomposition

$$M = M_2(e_1) \oplus M_1(e_1) \oplus M_0(e_1),$$

we have  $M_2(e_1) = N_2(e_1)$  and  $M_0(e_1) = N_0(e_1)$ . Pick  $x \in M_1(e_1)$ . Since  $e_1 \perp e_j$  ( $j = 2, 3$ ) we have  $M_1(e_1) \cap M_2(e_j) = \{0\}$  for every  $j = 2, 3$ . Therefore

$$x = P_1(e_2)(x) + P_0(e_2)(x),$$

where  $P_0(e_2)(x) \in M_0(e_2) = N_0(e_2) \subseteq N$  and  $P_1(e_2)(x) \in P_1(e_2)(N_1(e_1))$ . Since

$$\begin{aligned} \frac{1}{2}P_0(e_2)(x) + \frac{1}{2}P_1(e_2)(x) &= \frac{1}{2}x = \{e_1, e_1, x\} \\ &= \{e_1, e_1, P_0(e_2)(x)\} + \{e_1, e_1, P_1(e_2)(x)\}, \end{aligned}$$

it follows from Pierce rules that

$$\frac{1}{2}P_1(e_2)(x) = \{e_1, e_1, P_1(e_2)(x)\},$$



and hence  $P_1(e_2)(x) \in M_1(e_1) \cap M_1(e_2)$ . The condition  $e_1 \perp e_2$  leads us to  $\{e_1 + e_2, e_1 + e_2, P_1(e_2)(x)\} = P_1(e_2)(x)$ , which means that

$$P_1(e_2)(x) \in M_2(e_1 + e_2) = N_2(e_1 + e_2) \subseteq N.$$

We have therefore shown that  $x = P_1(e_2)(x) + P_0(e_2)(x) \in N$ , which implies that  $M_1(e_1) \subseteq N$  and consequently  $M = N$ . This concludes the proof.  $\square$

Let us recall that a  $C^*$ -algebra is reflexive if and only if it is finite dimensional (cf. [32, Proposition 2]). Consequently, a  $C^*$ -algebra has finite rank if and only if it is finite dimensional. It is further known that a  $C^*$ -algebra  $A$  has rank one if, and only if,  $A = \mathbb{C}1$ . In particular, the result established by Robertson in [30, Theorem 6] (see Theorem 2) is a direct consequence of our last theorem.

**Corollary 15.** *Let  $M$  be an infinite dimensional von Neumann algebra. Let  $N$  be a Čebyšev von Neumann subalgebra of  $M$ . Then  $N = \mathbb{C}1$  or  $M = N$ .  $\square$*

We have already seen that, for each natural  $n$ , we can find a complex Hilbert space (of dimension 2) which is a Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple having rank  $n$ . It is natural to ask whether we can find a precise description of those complex Hilbert spaces which are Čebyšev  $JBW^*$ -subtriples of a  $JBW^*$ -triple. Another general question that remains open in this paper is the following:

*Problem 16.* Determine the Čebyšev  $JB^*$ -subtriples of a general  $JB^*$ -triple.

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